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AN ERROR ANALYSIS FOR THE FINITE ELEMENT METHOD APPLIED TO CONV--ETC(U)

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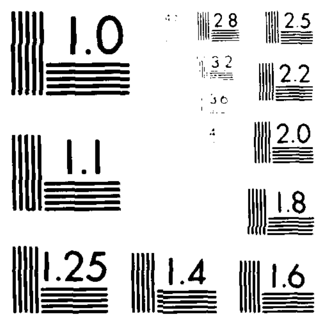

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AN ERROR ANALYSIS FOR THE FINITE ELEMENT METHOD APPLIED TO CONVECTION DIFFUSION PROBLEMS

by

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AN ERROR ANALYSIS FOR THE FINITE ELEMENT METHOD

APPLIED TO CONVECTION DIFFUSION PROBLEMS

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ABSTRACT

This paper analyzes the finite element method applied to a convection diffusion model problem. Linear elements are used for the trial space. The error is measured in a norm closely related to the L_p norm. When the test space is composed of linear elements with parabolic upwinding, the method is shown to be optimal when the input data is piecewise smooth -- a condition which is usually observed in practice. Without these smoothness assumptions, the method is shown to be non-optimal, even if the class of test spaces is extended to include any elements which have a shape independent of the mesh size.

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INTRODUCTION

A large amount of attention has recently been focused on the application of finite element methods to singularly perturbed boundary value problems. These problems arise, for example, in convection diffusion equations for fluid mechanics in which the convective term dominates. Finite element methods which employ the standard piecewise polynomial test and trial spaces lead to solutions having spurious oscillations unless the mesh size h is excessively small. A popular way to alleviate this problem is to "upwind" the test space. In the case of linear elements this can be done by adding a quadratic term, multiplied by some parameter α, to each piecewise linear basis function of the test space (see e.g. [5], [10]-[12], [15]-[17]). Heinrich, et al. [11], [12], Christie, et al. [6] and others have displayed the optimal α, which is determined in such a way so that for the problem

$$-\varepsilon u''+u' = 0 , \quad u(0) = 0 , \quad u(1) = 1 ,$$

the approximate solution agrees with the exact solution at the nodes. Griffiths [10] selects the parameter α in order to obtain the best quasi-optimal type estimate of the form

$$||u-u_h|| \leq c(\varepsilon,h) \inf_{w\in S_h} ||u-w|| \quad ,$$

where u is the exact solution, u_h is the approximate solution, S_h is the trial space, and $||\cdot||$ is the L_2 norm of the derivative. However, the constant c is not bounded uniformly in ε and h; for example, if ε is very small with respect to h, $c(\varepsilon,h)$ will increase with rate h^{-1} as $h \to 0$.

Upwinding of the test space can also be done through the use of L-spline basis functions. Methods using these spaces are studied in Hemker [13], de Groen [7], and Hemker and de Groen [8], but a norm is used to measure the error between the exact solution u and its piecewise linear approximation in which the error cannot be small unless $h << \varepsilon$.

The analysis in this paper is done specifically for the model problem

$$-\varepsilon u'' + u' = f \ , \tag{1.1}$$

$$u(0) = u(1) = 0 \ .$$

In section 2 we present the general theory for obtaining quasi-optimal estimates, as established in Babuska, Aziz [1]. Mesh dependent norms for the model problem, analogous to those presented in [2], are defined in section 3. We will be measuring the error in a norm similar to the L_p norm, which is appropriate in our case because it allows for approximability when using a piecewise linear trial space. In section 4 a quasi-optimal result of the form

$$||u-u_h|| \leq c \inf_{w \in S_h} ||u-w|| \quad , \tag{1.2}$$

where c is independent of ε and h, is proven when the test space is composed of L-spline basis functions. However, it is also shown that for a large class of test functions, it is in general impossible to obtain a result of the form (1.2). In particular, quasi-optimality cannot be proven with the use of quadratically upwinded elements for any choice of the parameter α unless we restrict ourselves to the case of "reasonable" inputs f. This is shown in our main result, presented in section 5, which says that if f is piecewise smooth, and α is chosen to be the optimal one introduced by e.g. Christie et al. [6], and

Zienkiewicz and Heinrich [17], then quasi-optimality is obtained.

2. SOME ABSTRACT RESULTS

In this section we review two crucial results concerning variationally formulated boundary value problems and finite element approximations.

Theorem 2.1. Let $K_{1,\Delta}$ and $K_{2,\Delta}$ be two reflexive Banach spaces, indexed by a parameter Δ with Δ varying over some index set, with norms $||\cdot||_{1,\Delta}$ and $||\cdot||_{2,\Delta}$ respectively, and let B be a bilinear form on $K_{1,\Delta} \times K_{2,\Delta}$. We suppose the following are satisfied:

(2.1) $$|B_\Delta(u,v)| \le C_1 ||u||_{1,\Delta} ||v||_{2,\Delta} \quad \text{for all} \quad u \in K_{1,\Delta}, \quad v \in K_{2,\Delta} \;,$$

(2.2) $$\inf_{\substack{u \in K_{1,\Delta} \\ ||u||_{1,\Delta}=1}} \; \sup_{\substack{v \in K_{2,\Delta} \\ ||v||_{2,\Delta}=1}} |B_\Delta(u,v)| \ge C_2 > 0 \;,$$

and

(2.3) $$\sup_{u \in K_{1,\Delta}} |B_\Delta(u,v)| > 0 \;, \quad \text{for each} \quad 0 \ne v \in K_{2,\Delta} \;,$$

where C_1 and C_2 are positive constants, possibly depending on Δ. Then if $f \in (K_{2,\Delta})'$, there exists a unique solution $u \in K_{1,\Delta}$ to the problem

(2.4) $$B_\Delta(u,v) = f(v) \qquad \forall v \in K_{2,\Delta} \;.$$

Moreover, u satisfies

$$||u||_{1,\Delta} \leq C_2^{-1} ||f||_{K'_{2,\Delta}} \quad .$$

If the bilinear form $B_\Delta(\cdot,\cdot)$ satisfies the assumptions (2.1), (2.2), and (2.3), B_Δ is said to be a (C_1,C_2)-proper bilinear form over the space $K_{1,\Delta} \times K_{2,\Delta}$. It should be noted that (2.2) and (2.3) can be shown to be equivalent to

$$(2.2)^* \qquad \inf_{\substack{v \in K_{2,\Delta} \\ ||v||_{2,\Delta}=1}} \ \sup_{\substack{u \in K_{1,\Delta} \\ ||u||_{1,\Delta}=1}} |B_\Delta(u,v)| \geq C_2^* > 0 \quad ,$$

and

$$(2.3)^* \qquad \sup_{v \in K_{2,\Delta}} |B_\Delta(u,v)| > 0 \ , \quad \forall 0 \neq u \in K_{1,\Delta} \quad .$$

This observation will be specifically used in this paper.

Since we will be studying finite element approximations to u, we let $S_{1,\Delta}$ and $S_{2,\Delta}$ be finite dimensional subspaces of $K_{1,\Delta}$ and $K_{2,\Delta}$, respectively. Clearly condition (2.1) holds on $S_{1,\Delta} \times S_{2,\Delta}$ with the same constant C_1. We will be invoking the following theorem concerning the finite element solution u_h.

Theorem 2.2. Suppose B_Δ is (C_1,C_2')-proper over $S_{1,\Delta} \times S_{2,\Delta}$ furnished with the norms $||\cdot||_{1,\Delta}$ and $||\cdot||_{2,\Delta}$, respectively. Let $u \in K_{1,\Delta}$, and let $u_h \in S_{1,\Delta}$ be the unique solution to $B_\Delta(u_h,v) = B_\Delta(u,v)$ for all $v \in S_{2,\Delta}$. Then

$$||u-u_h||_{1,\Delta} \leq (1 + \frac{C_1}{C_2'}) \inf_{w \in S_{1,\Delta}} ||u-w||_{1,\Delta}$$

For the proof of Theorems 2.1 and 2.2, see e.g. [1].

3. MESH DEPENDENT NORMS AND SPACES

In this section we define the various norms, spaces, and bilinear forms used throughout this paper. The norms introduced here are analogous to those defined in [2].

Throughout the paper $H_p^k(I)$, $k = 0,1,\ldots,$ $1 \le p \le \infty$, will denote the usual Sobolev space on the interval I in R^1 consisting of functions with k derivatives in $L_p(I)$. On this space we have the usual norms given by

$$(3.1)\qquad ||u||_{k,p,I} = \begin{cases} \left(\sum_{j=0}^{k} \int_I |u^{(j)}(x)|^p dx\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sum_{j=0}^{k} \operatorname{ess\,sup} |u^{(j)}|, & p = \infty. \end{cases}$$

$\overset{\bullet}{H}{}_p^1(I)$ denotes the subspace of $H_p^1(I)$ of functions that vanish at the endpoints of I. Note that $H_p^o = L_p$.

Let $\Delta = \{0 = x_o < x_1 < \ldots x_N = 1\}$, where $N = N(\Delta)$, be an arbitrary mesh on the interval $I = [0,1]$. Let $h_j = x_j - x_{j-1}$, $I_j = (x_{j-1}, x_j)$, $j = 1,\ldots N$, $\rho_j = (h_j + h_{j+1})/2$, $j = 1,\ldots N-1$, and $h(\Delta) = \max_j h_j$.

We now define the space $H_{p,\Delta}^o$, $1 \le p \le \infty$, to be the completion of $\overset{\circ}{H}{}_p^1(I)$ furnished with the norm

$$(3.2)\qquad ||u||_{H_{p,\Delta}^o} = \begin{cases} \left(\int_0^1 |u|^p dx + \sum_{j=1}^{N-1} \rho_j |u(x_j)|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ ||u||_{L_\infty(I)}, & p = \infty. \end{cases}$$

The space $H_{p,\Delta}^o$ can be easily identified with $L_p \oplus \mathbb{R}^{N-1}$, that is,

$u = (\tilde{u}, d_1, \ldots, d_{N-1}) \in H^o_{p,\Delta} = L_p \oplus \mathbb{R}^{N-1}$, and

$$(3.3) \qquad \|u\|_{H^o_{p,\Delta}} = \begin{cases} \{\|\tilde{u}\|^p_{L_p(I)} + \sum_{j=1}^{N-1} \rho_j |d_j|^p\}^{\frac{1}{p}} , & 1 \leq p < \infty , \\ \max\{\|\tilde{u}\|_{L_\infty(I)}, |d_j|\} , & p = \infty . \end{cases}$$

In consistency with our definition, we say $u \in H^o_{p,\Delta} \cap \overset{\circ}{H}{}^1_p(I)$ if $\tilde{u} \in \overset{\circ}{H}{}^1_p(I)$ and $d_j = \tilde{u}(x_j)$, $j = 1, \ldots N-1$.

Let us now define $H^2_{p,\Delta} = \{v \in \overset{\circ}{H}{}^1_p(I) : v|_{I_j} \in H^2_p(I_j), \ j = 1, \ldots N\}$. We will equip this space with a norm to be defined later.

On $H^o_{p,\Delta} \times H^2_{q,\Delta}$, where $\frac{1}{p} + \frac{1}{q} = 1$, we define a bilinear form $B_\Delta(\cdot,\cdot)$ by

$$(3.4) \qquad B_\Delta(u,v) = \sum_{j=1}^{N} \int_{I_j} \tilde{u}(-\varepsilon v'' - v')dx - \sum_{j=1}^{N-1} \varepsilon d_j J(v'(x_j)) .$$

where $J(v'(x_j)) = v'(x_j+0) - v'(x_j-0)$ and $v'(x_j \pm 0) = \lim_{x \to x_j^\pm} v'(x)$. These limits are well defined because $v|_{I_k} \in H^2_q(I_k)$ for each k . Now we will furnish the space $H^2_{p,\Delta}$ with the norm $|||\cdot|||$, defined by

$$(3.5) \qquad |||v||| = \sup_{u \in H^o_{q,\Delta}} \frac{|B_\Delta(u,v)|}{\|u\|_{H^o_{q,\Delta}}} .$$

It is evident that the triangle inequality and linearity are satisfied by $|||\cdot|||$. We must show positive definiteness. To do that we make use of the following identity for $v \in H^2_{p,\Delta}$:

$$(3.6)\qquad v(x) = \int_0^x (L^*v)(t)(1-e^{-\frac{x-t}{\varepsilon}})dt - \sum_{\substack{1\le i\le N-1\\ x_i\le x}} (e^{-\frac{(x-x_i)}{\varepsilon}}-1)\varepsilon J(v'(x_i)) +$$

$$+ \frac{1-e^{-\frac{x}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\left[\sum_{i=1}^{N-1} (e^{\frac{(x_i-1)}{\varepsilon}}-1)\varepsilon J(v'(x_i)) - \int_0^1 (L^*v)(t)(1-e^{-\frac{1-x}{\varepsilon}})\right]$$

where

$$(3.7)\qquad (L^*v)\big|_{I_j} = (\varepsilon v''+v')\big|_{I_j}\ ,\qquad j = 1,\dots,N\quad .$$

For $v\in H^2_{p,\Delta}$, select $u_o = (\tilde{u}_o,d_1,d_2,\dots d_{N-1})$ such that $\tilde{u}_o|_{I_j} = \mathrm{sgn}(L^*v)|_{I_j}$, and $d_j = -\,\mathrm{sgn}J(v'(x_j))$, $j-1,\dots,N-1$, then

$$(3.8)\qquad |B(u_o,v)| = \sum_{j=1}^{N}\int_{I_j} |L^*v|dx + \sum_{j=1}^{N-1}\varepsilon|J(v'(x_j))| \ge 0\quad .$$

If $B(u_o,v) = 0$ then $L^*v|_{I_k} = 0$ and $J(v'(x_j)) = 0$, and (3.6) yields $v = 0$. Therefore, $|||\cdot|||$ is positive definite, and hence, a norm.

Let us now introduce another norm. For any $v\in H^2_{p,\Delta}$ define $||\cdot||_{H^2_{p,\varepsilon,\Delta}}$ by

$$(3.9)\qquad ||v||_{H^2_{p,\varepsilon,\Delta}} = \begin{cases} \left[\sum_{j=1}^{N}\int_{I_j} |\varepsilon v''+v'|^p dx + \sum_{j=1}^{N-1}\varepsilon^p|J(v'(x_j))|^p\rho_j^{1-p}\right]^{\frac{1}{p}}\ , & \\ \qquad\qquad \text{if}\quad 1\le p<\infty\ , & \\ \max_{1\le j\le N}||\varepsilon v''+v'||_{\infty,I_j} + \max_{1\le j\le N-1}\varepsilon|J(v'(x_j)|\rho^{-1}\ , & \\ \qquad\qquad \text{if}\quad p=\infty\quad . & \end{cases}$$

We shall now prove that the norms $||\cdot||_{H^2_{p,\varepsilon,\Delta}}$ and $|||\cdot|||$ are equal for $1 \le p < \infty$ and equivalent when $p = \infty$.

Lemma 3.1. Let $v \in H^2_{p,\Delta}$, then

(3.10) $$||v||_{H^2_{p,\varepsilon,\Delta}} = |||v||| \qquad 1 \le p < \infty ,$$

and

$$\frac{1}{2}||v||_{H^2_{\infty,\varepsilon,\Delta}} \le |||v||| \le ||v||_{H^2_{\infty,\varepsilon,\Delta}} , \quad p = \infty .$$

Proof. For $1 < p < \infty$, a straightforward application of Hölder's inequality yields

$$|B_\Delta(u,v)| \le \left(\int_0^1 |\tilde{u}|^q dx\right)^{\frac{1}{q}} \left(\sum_{j=1}^{N} \int_{I_j} |\varepsilon v''+v'|^p dx\right)^{\frac{1}{p}}$$

$$+ \left(\sum_{j=1}^{N-1} \rho_j |d_j|^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{N-1} \varepsilon^p |J(v'(x_j))|^p \rho_j^{1-p}\right)^{\frac{1}{p}}$$

$$\le \left[\int_0^1 |\tilde{u}|^q dx + \sum_{j=1}^{N-1} \rho_j |d_j|^q\right]^{\frac{1}{q}} \left[\sum_{j=1}^{N} \int_{I_j} |\varepsilon v''+v'|^p dx\right.$$

$$\left. + \sum_{j=1}^{N-1} \varepsilon^p |J(v'(x_j))|^p \rho_j^{1-p}\right]^{\frac{1}{p}} = ||u||_{H^o_{q,\Delta}} ||v||_{H^2_{p,\varepsilon,\Delta}} ,$$

and thus

(3.11) $$|||v||| \le ||v||_{H^2_{p,\varepsilon,\Delta}} .$$

That 3.11 holds when $p = 1$ and $p = \infty$ follows directly from Hölder's inequality.

For the inequality in the other direction, let $v \in H^2_{p,\Delta}$, $1 \le p < \infty$, be given. Select $u_v \equiv (\tilde{u}, d_1, \ldots d_{N-1}) \in H^o_{q,\Delta}$ as follows:

$$\tilde{u} = -|L^*v|^{p-1}\operatorname{sgn}(L^*v) \quad ,$$

$$d_j = -\varepsilon^{p-1}|J(v'(x_j))|^{p-1}\rho_j^{1-p}\operatorname{sgn}(J(v'(x_j)))$$

$$\text{for} \quad 1 \le j \le N-1 \quad .$$

Then $$||u_v||^q_{H^o_{q,\Delta}} = ||v||^p_{H^2_{p,\varepsilon,\Delta}} \quad , \quad 1 < p < \infty \quad ,$$

$$||u_v||_{H^o_{\infty,\Delta}} = 1 \quad , \qquad p=1 \quad ,$$

and $$B_\Delta(u_v, v) = ||v||^p_{H^2_{p,\varepsilon,\Delta}} \quad , \quad 1 \le p < \infty \quad ,$$

This immediately yields

$$|||v||| \ge ||v||_{H^2_{p,\varepsilon,\Delta}} \quad , \quad 1 \le p < \infty \quad .$$

For $p = \infty$, let L be the index such that

$$||\varepsilon v''+v'||_{L_\infty(I_L)} = \max_{1\le j\le N} ||\varepsilon v''+v'||_{L_\infty(I_j)} \quad ,$$

and let J be the index such that

$$\varepsilon|J(v'(x_J))|\rho_J^{-1} = \max_{1\le j\le N-1} \varepsilon|J(v'(x_j))|\rho_j^{-1} \quad .$$

Let $\eta > 0$ be given and define

$$E_\eta = \{x \in I_L : |(\varepsilon v'' + v')(x)| > ||\varepsilon v'' + v'||_{L_\infty(I_L)} - \eta\} \quad .$$

Then $m(E_\eta) > 0$, where $m(A)$ is the Lebesgue measure of A.

For $v \in H^2_{\infty,\Delta}$ define $u_N = (\tilde{u}, d_1, \ldots d_{N-1})$ as follows:

$$\tilde{u} = -\chi_{E_\eta}(m(E_\eta))^{-1}\mathrm{sgn}(L^*v) \quad , \tag{3.12}$$

$$d_j = \begin{cases} \rho_J^{-1}\mathrm{sgn}(J(v'(x_J))), & \text{if } j = J \ , \\ 0 \ , & j = 1, \ldots J-1, J+1, \ldots N-1 \ , \end{cases}$$

where χ_A denotes the characteristic function of the set A . Then $||u_v||_{H^o_{1,\Delta}} = 2$ and

$$|||v||| \geq (||\varepsilon v'' + v'||_{L_\infty(I_L)} - \eta + \varepsilon|J(v'(x_J))|\rho_J^{-1})/2 \quad .$$

Since η was arbitrary,

$$|||v||| \geq \frac{||v||_{H^2_{\infty,\varepsilon,\Delta}}}{2} \quad \underline{1/} \quad ,$$

which concludes the proof.

1/ That this estimate is sharp can be seen by taking, for example, $\Delta = \{0, \frac{1}{2}, 1\}$, $\varepsilon = \frac{1}{4}$, $v(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2}, \\ 1-x, & \frac{1}{2} \leq x \leq 1 \end{cases}$. Then $||v||_{H^2_{\infty,\varepsilon,\Delta}} = 2$, but $|||v||| = 1$.

However, any v with $L^*v|_{I_j} = 0$ for $j=1,\ldots N$ has $||v||_{H^2_{\infty,\varepsilon,\Delta}} = |||v|||$.

We now prove an imbedding theorem.

Lemma 3.2. Let $v \in H^2_{p,\varepsilon,\Lambda} = (H^2_{p,\Delta}, ||\cdot||_{H^2_{p,\varepsilon,\Delta}})$, then $v \in L_\infty(I) \cap H^1_p(I)$ with

$$(3.13) \qquad ||v||_{L_\infty(I)} \le 4||v||_{H^2_{p,\varepsilon,\Delta}}$$

and

$$(3.14) \qquad ||v||_{1,p,I} \le \left\{ 5 + \varepsilon^{(\frac{1}{p}-1)} \left[1 + \frac{2(1-e^{-\frac{p}{\varepsilon}})^{\frac{1}{p}}}{p^{\frac{1}{p}}(1-e^{-\frac{1}{\varepsilon}})} \right] \right\} ||v||_{H^2_{p,\varepsilon,\Delta}} ,$$

for $1 \le p \le \infty$.

Furthermore, if we assume $0 < \varepsilon \le \varepsilon_o$, then

$$(3.15) \qquad ||v||_{1,p,I} \le c\varepsilon^{(\frac{1}{p}-1)} ||v||_{H^2_{p,\varepsilon,\Delta}} = c\varepsilon^{-\frac{1}{q}} ||v||_{H^2_{p,\varepsilon,\Delta}} ,$$

$$1 \le p \le \infty, \ \frac{1}{p} + \frac{1}{q} = 1 ,$$

where c is independent of ε , p , and Λ .

Proof: We use (3.6) for $v(x)$ and write

$$v(x) = w_1(x) + w_2(x) + w_3(x) , \quad \text{where}$$

$$w_1(x) = \int_0^x ((L^*v)(t))(1-e^{-\frac{(x-t)}{\varepsilon}})dt ,$$

$$w_2(x) = -\sum_{\substack{1\le i\le N-1\\ x_i\le x}} \left(e^{-\frac{(x-x_i)}{\varepsilon}} - 1\right)\varepsilon J(v'(x_i))$$

and

$$w_3(x) = -\left(\frac{1-e^{-\frac{x}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)(w_1(1) + w_2(1)) \ .$$

By inspection, and an application of Hölder's inequality, we obtain

$$|w_1(x)| \le \int_0^1 |L^*v|\,dt \le ||v||_{H^2_{p,\varepsilon,\Delta}} \ ,$$

and

$$|w_2(x)| \le \sum_{j=1}^{N-1} \varepsilon|J(v'(x_j))| = \sum_{j=1}^{N-1} \varepsilon|J(v'(x_j))|\rho_j^{\frac{1}{p}-1}\rho_j^{\frac{1}{q}} \le ||v||_{H^2_{p,\varepsilon,\Delta}} \ ,$$

$$1 \le p \le \infty \ .$$

Since $|w_3(x)| \le |w_1(1)| + |w_2(1)| \le 2||v||_{H^2_{p,\varepsilon,\Delta}}$, (3.12) is proven. Also

$$(3.16) \qquad ||w_3'||_{L_p(I)} \le \frac{2}{\varepsilon(1-e^{-1/\varepsilon})}\left[\int_0^1 e^{-px/\varepsilon}dx\right]^{1/p} ||v||_{H^2_{p,\varepsilon,\Delta}}$$

$$= \frac{2(1-e^{-\frac{p}{\varepsilon}})^{\frac{1}{p}}}{p^{\frac{1}{p}}(1-e^{-\frac{1}{\varepsilon}})}\,\varepsilon^{\frac{1}{p}-1}\,||v||_{H^2_{p,\varepsilon,\Delta}} \ , \qquad 1 \le p < \infty \ .$$

For $p = \infty$ we have $||w_3'||_{L_\infty(I)} \leq \frac{2\varepsilon^{-1}}{1-e^{-\frac{1}{\varepsilon}}} ||v||_{H^2_{\infty,\varepsilon,\Delta}}$, which is the limiting estimate for $p < \infty$.

Next,

$$w_2'(x) = \sum_{\substack{1\leq i\leq N-1 \\ x_i\leq x}} e^{-\frac{x-x_i}{\varepsilon}} J(v'(x_j)) = \sum_{i=1}^{N-1} z_i(x) \quad ,$$

where

$$z_i(x) = \begin{cases} 0 \ , & \text{if} \quad x_i > x \ , \\ e^{-\frac{(x-x_i)}{\varepsilon}} J(v'(x_i)) \ , & \text{if} \quad x_i \leq x \ . \end{cases}$$

Therefore, we have

$$(3.17) \qquad ||w_2'||_{L_p(I)} \leq \sum_{i-1}^{N-1} ||z_i||_{L_p(I)} \leq \sum_{i=1}^{N-1} (\frac{\varepsilon}{p})^{\frac{1}{p}} |J(v'(x_i))| \leq$$

$$\leq \varepsilon^{\frac{1}{p}-1} \sum_{i=1}^{N-1} \varepsilon |J(v'(x_i))| \rho_i^{\frac{1}{p}-1} \rho_i^{\frac{1}{q}} \leq \varepsilon^{\frac{1}{p}-1} ||v||_{H^2_{p,\varepsilon,\Delta}} \quad ,$$

$$1 \leq p \leq \infty \quad .$$

Finally, $w_1'(x) = \int_0^x (L^*v)(t) \frac{1}{\varepsilon} e^{-(\frac{x-t}{\varepsilon})} dt = \int_0^1 (L^*v)(t) \frac{1}{\varepsilon} \phi(\frac{x-t}{\varepsilon}) dt$

where

$$\phi(x) = \begin{cases} e^{-x}, & \text{if} \quad x \geq 0 \ , \\ 0 \ , & \text{if} \quad x < 0 \quad . \end{cases}$$

Since $L^*v \in L_p(I)$ and $\frac{1}{\varepsilon}\,\phi(\frac{x}{\varepsilon}) \in L_1(I)$, Young's inequality[2/] implies that $w_1' \in L_p(I)$ and

(3.18) $$\|w_1'\|_{L_p(I)} \le \|L^*v\|_{L_p(I)} \|\tfrac{1}{\varepsilon}\,\phi(\tfrac{x}{\varepsilon})\|_{L_p(I)}$$

$$\le \|L^*v\|_{L_p(I)} \le \|v\|_{H^2_{p,\varepsilon,\Delta}} \quad , \quad 1 \le p \le \infty \quad .$$

Combining (3.16), (3.17), and (3.18) with (3.13) yields (3.14).

Lemma 3.1, equation (3.5) and a simple verification of (2.3) imply that $B_\Delta(u,v)$ is (1,1)-proper on $H^o_{p,\Delta} \times H^2_{q,\varepsilon,\Delta}$ for $1 < p \le \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $(1,\frac{1}{2})$-proper on $H^o_{1,\Delta} \times H^2_{\infty,\varepsilon,\Delta}$. Also, for $1 < p < \infty$, the spaces $H^o_{p,\Delta}$ and $H^2_{p,\varepsilon,\Delta}$ are reflexive, and so in this case we may apply Theorem 2.1 which leads to the existence of either the solution u, for the problem

$$B_\Delta(u,v) = f(v), \quad \forall\, v \in H^2_{p,\varepsilon,\Delta} \quad ,$$

or the solution v of the adjoint problem

$$B_\Delta(u,v) = f(u), \quad \forall\, u \in H^o_{p,\Delta} \quad .$$

Existence for the cases $p = 1$ or $p = \infty$ does not follow from Theorem 2.1 directly, but has to be dealt with separately.

[2/] Young's inequality states that for $1 \le p \le \infty$, if $s \in L^p(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, then $h = s*g$ exists a.e. and belongs to $L^p(\mathbb{R}^n)$ and

$$\|h\|_{L_p} \le \|s\|_{L_p} \|g\|_{L_1} \quad .$$

Of course to apply this result to our case, we should extend L^*v by zero to all of $\mathbb{R}$.

4. THE FINITE DIMENSIONAL SPACES OF FINITE ELEMENTS

In this section we will analyze the behavior of the bilinear form (3.4) on finite dimensional spaces $S_{1,\Delta} \times S_{2,\Delta}$, $S_{1,\Delta} \subset H^{o}_{p,\Delta}$, $S_{2,\Delta} \subset H^{2}_{q,\varepsilon,\Delta}$. The space of trial functions $S_{1,\Delta}$ will be the usual space of linear elements

(4.1) $$S_{1,\Delta} = \{\tilde{u},d_1,\ldots,d_{N-1}:\tilde{u}\in \mathring{H}^1_2,\tilde{u}|_{I_j} \text{ is linear, } j=1,\ldots N,\ d_j=\tilde{u}(x_j),\ j=1,\ldots N-1\}.$$

However, for the test space $S_{2,\Delta}$ we will allow various other possibilities, especially

(4.2) $$S^{E}_{2,\Delta} = \{v\in \mathring{H}^1_2 : L^*v|_{I_j} = 0,\ j=1,\ldots N\} ,$$

and

(4.3) $$S^{\alpha}_{2,\Delta} = \{v\in \mathring{H}^1_2 : v = \sum_{j=1}^{N-1} v_j\psi^{\alpha}_j(x)\} ,$$

where

$$\psi^{\alpha}_j(x) = \begin{cases} \dfrac{1}{h_j}(x-x_{j-1}) + \dfrac{3\alpha_j}{h_j^2}(x-x_{j-1})(x_j-x), & \text{for } x\in I_j , \\[2ex] \dfrac{1}{h_{j+1}}(x_{j+1}-x) - \dfrac{3\alpha_{j+1}}{h_{j+1}^2}(x-x_j)(x_{j+1}-x), & \text{for } x\in I_{j+1} , \end{cases}$$

and $\alpha_j = \alpha_j(\varepsilon,h_j)$, i.e., in general α_j will depend on ε and h_j .

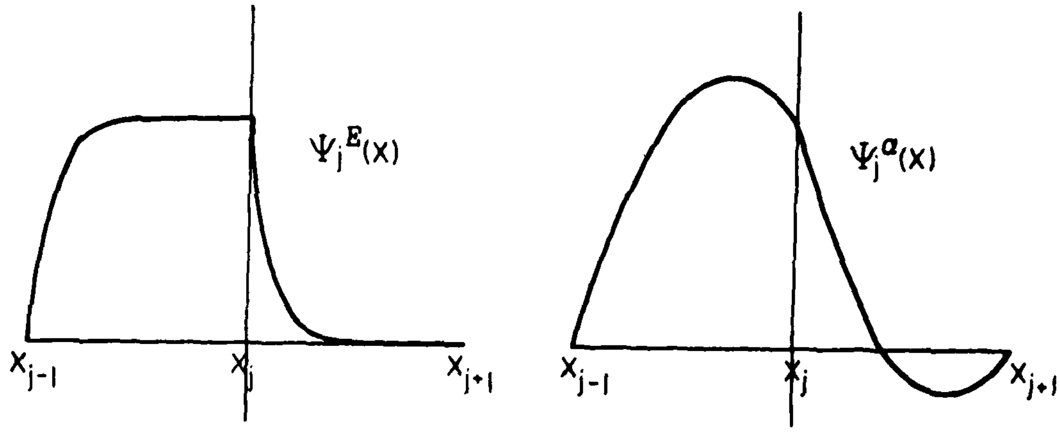

Figure 1: The basis functions $\psi_j^E \in S_{2,\Delta}^E$ and $\psi_j^\alpha \in S_{2,\Delta}^\alpha$.

Figure 1 shows the shapes of the basis functions for the spaces $S_{2,\Delta}^E$ and $S_{2,\Delta}^\alpha$.

Remark: (4.1) is the space of L*-splines and (4.2) was introduced in [5], [10]-[12], [15]-[17]. For other various analyses of the use of L*-splines, we refer the reader e.g. to [7]-[9], [13].

We will now study the inf-sup condition of the bilinear form $B_\Delta(u,v)$ on $S_{1,\Delta} \times S_{2,\Delta}$, when $S_{2,\Delta} = S_{2,\Delta}^E$ defined by (4.2).

Theorem 4.1. Let $S_{1,\Delta} \subset H_{p,\Delta}^o$, $S_{2,\Delta}^E \subset H_{q,\varepsilon,\Delta}^2$, be defined by (4.1) and (4.2) respectively, with $\frac{1}{p} + \frac{1}{q} = 1$ *and* $1 \leq p \leq \infty$. Then the bilinear form B_Δ defined by (3.4) is (C_1,C_2')-proper with $C_1 = 1$, and $C_2' = 2^{-1/p}$.

Proof. Given $v \in S_{2,\Delta}^E$, select $u_v \in S_{1,\Delta}$ such that

$$u_v(x_j) = -\varepsilon^{\frac{q}{p}} |J(v'(x_j))|^{\frac{q}{p}} \rho_j^{-\frac{q}{p}} \operatorname{sgn} J(v'(x_j))$$

$$j = 1,\ldots,n-1$$

for $1 < p \le \infty$ (or $1 \le q < \infty$) .

Then we have

$$\text{(4.4)} \qquad B_\Delta(u_v,v) = \sum_{j=1}^{N-1} \varepsilon^{1+\frac{q}{p}} |J(v'(x_j))|^{(1+q/p)} \rho_j^{-q/p}$$

$$= \sum_{j=1}^{N-1} \varepsilon^q |J(v'(x_j))|^q \rho_j^{1-q} = ||v||^q_{H^2_{q,\varepsilon,\Delta}} .$$

For any piecewise linear u we have

$$\int_o^1 |u|^p dx = \sum_{j=1}^{N} \int_{I_j} |u|^p dx \le \sum_{j=1}^{N} \frac{1}{2}(|u(x_{j-1})|^p + |u(x_j)|^p) h_j \le$$

$$\le \sum_{j=1}^{N-1} |u(x_j)|^p \rho_j \qquad 1 \le p < \infty \quad ,$$

and $||u||_{L_\infty(I)} = \max_{1 \le j \le N-1} |u(x_j)|$.

Therefore,

$$\text{(4.5)} \quad ||u_v||^p_{H^o_{p,\Delta}} \le 2\sum_{j=1}^{N-1} |u_v(x_j)|^p \rho_j \le 2\sum_{j=1}^{N-1} \varepsilon^q |J(v'(x_j))|^q \rho_j^{-q+1} = 2||v||^q_{H^2_{q,\varepsilon,\Delta}} ,$$

$1 \le p \le \infty$, and $||u_v||_{H^o_{\infty,\Delta}} = ||u_v||_{L_\infty(I)} = 1$.

Finally, (4.4) and (4.5) imply

$$\frac{B_\Delta(u_v,v)}{||u_v||_{H^o_{p,\Delta}}} = \frac{||v||^q_{H^2_{q,\varepsilon,\Delta}}}{||u_v||_{H^o_{p,\Delta}}} \ge 2^{-1/p} ||v||^{q-q/p}_{H^2_{q,\varepsilon,\Delta}} = 2^{-1/p} ||v||_{H^2_{q,\varepsilon,\Delta}} , \quad 1 < p \le \infty .$$

If $p = 1$, $q = \infty$ take $u_v(x_j) = d_j$, where d_j, $j = 1,\ldots,N-1$, is defined by (3.12). Then $\frac{B_\Delta(u_v,v)}{||u_v||_{H^o_{p,\Delta}}} = \frac{1}{2}||v||_{H^2_{\infty,\varepsilon,\Delta}}$, which corresponds to the above estimate when $p = 1$. Hence, Theorem 4.1 follows with $C_2' = 2^{-\frac{1}{p}}$, $1 \le p \le \infty$. Obviously $C_1 = 1$, and condition (2.3)' is also easy to check.

This leads to the following theorem.

Theorem 4.2. Let $S_{1,\Delta}$ and $S^E_{2,\Delta}$ be the spaces defined by (4.1) and (4.2), respectively and $1 \le p \le \infty$. If $u \in H^o_{p,\Delta}$, and we denote by $u^E_h \in S_{1,\Delta}$ the function such that

$$B_\Delta(u^E_h,v) = B(u,v) \qquad \forall v \in S^E_{2,\Delta} \quad ,$$

then

$$(4.6) \qquad ||u^E_h-u||_{H^o_{p,\Delta}} \le (1+2^{\frac{1}{p}}) \inf_{\omega \in S_{1,\Delta}} ||u-\omega||_{H^o_{p,\Delta}} \quad .$$

Proof. Theorem (4.2) follows immediately from Theorem 4.1 and Theorem 2.2. The next theorem shows that $(u-u^E_h)(x_j) = 0$, for $j = 1,\ldots,N-1$.

Theorem 4.3. Let $u \in H^o_{p,\Delta}$, $1 \le p \le \infty$, and $u^E_h \in S_{1,\Delta} \in H^o_{p,\Delta}$, with

$$(4.7) \qquad B(u^E_h,v) = B(u,v) \ , \qquad \forall v \in S^E_{2,\Delta} \quad .$$

Then

$$(4.8) \qquad u^E_h(x_i) - d_i = 0 \ , \qquad i = 1,\ldots,N-1 \quad ,$$

with $u = (\tilde{u},d_1,\ldots d_{N-1})$.

Proof. Let $1 < p < \infty$. By Theorem 2.1, there exists $v_i \in H^2_{p,\varepsilon,\Delta}$ such that

$$(4.9) \qquad B_\Delta(u,v_i) = d_i \qquad \forall u \in H^o_{q,\Delta} \ , \quad \frac{1}{p} + \frac{1}{q} = 1 \ .$$

(4.9) yields $-\varepsilon v_i'' - v_i' = 0$, on every I_j , $J(v_i'(x_j)) = 0$, $j \neq i$, and $J(v_i'(x_i)) = \frac{1}{\varepsilon}$.

Because $v_i(x)$ is continuous, and $\varepsilon v_i'' + v_i' = 0$, on $(0,x_i)$ and $(x_i,1)$, v_i is the Green's function at $x = x_i$. Obviously $v_i \in H^2_{\infty,\varepsilon,\Delta}$ and $v_i \in S^E_{2,\Delta}$. This implies that (4.9) holds for $1 \leq q \leq \infty$. Thus (4.7) and (4.9) imply

$$B(u_h^E-u,v_i) = 0 = (u_h^E(x_i)-d_i) \quad ,$$

which finishes the proof.

Theorem 4.3 is a restatement of the well known fact that when the Green's functions at $x = x_i$, $1 \leq i \leq N-1$, belong to the test space, then the error at the nodal points is zero.

Now we turn to the study of the space $S_{2,\Delta}$ in a more general framework. Since $B_\Delta(u,v)$ depends only on the derivatives of the test functions, we let

$$(4.10) \qquad v' = \sum_{j=1}^{N} a_j \chi_j$$

where χ_j has support in I_j .

The situation is clear from Fig. 2

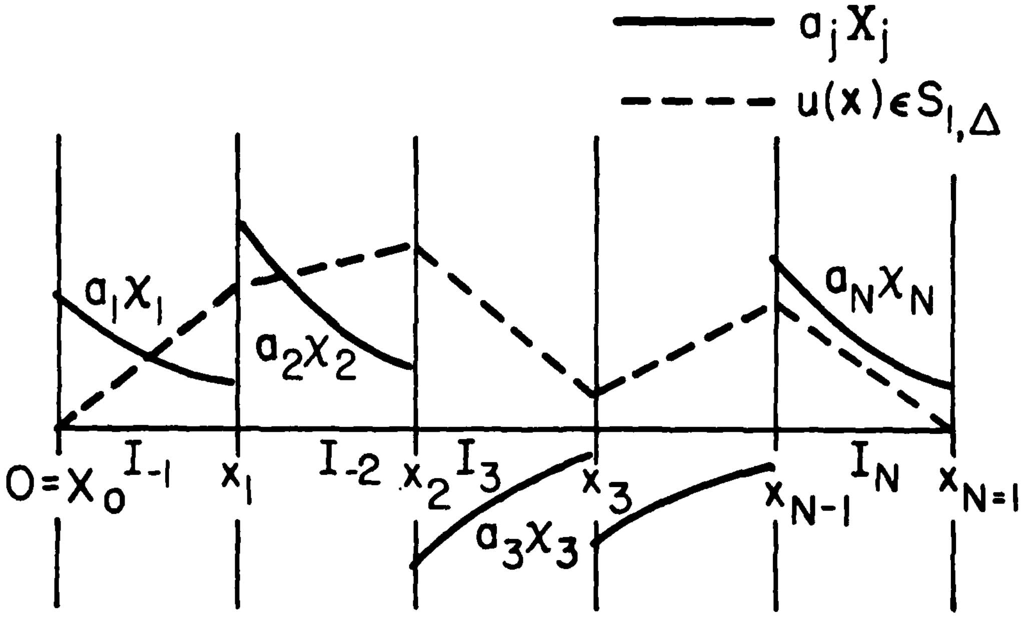

Figure 2: The graph of $v' = \sum_{j=1}^{N} a_j \chi_j$, where $v \in S_{2,\Delta}$, and $u \in S_{1,\Delta}$.

Obviously we have $\int_0^1 v' dx = 0$, because $v(0) = v(1) = 0$.

Let χ_j, $j = 1,\ldots,N$ have the following properties.

$$\text{(4.11) i)} \quad \int_{x_{j-1}}^{x_j} \chi_j(x) dx = \gamma_1^{[j]} h_j \quad ,$$

$$\text{ii)} \quad \int_{x_{j-1}}^{x_j} [\varepsilon \chi_j' + \chi_j] \frac{x_j - x}{h_j} dx = \gamma_2^{[j]} h_j \quad ,$$

$$\text{iii)} \quad \int_{x_{j-1}}^{x_j} [\varepsilon \chi_j' + \chi_j] \frac{x - x_{j-1}}{h_j} dx = \gamma_3^{[j]} h_j \quad ,$$

$$\text{iv)} \quad \int_{x_{j-1}}^{x_j} |\varepsilon \chi_j' + \chi_j|^p dx = \gamma_4^{[j]} h_j \quad ,$$

v) a) $\chi_j(x_{j-1}) = \gamma_5^{[j]}$,

b) $\chi_j(x_j) = \gamma_6^{[j]}$.

For $u \in S_{1,\Delta}$ with $u(x_j) = d_j$, $j=1,\ldots N-1$, and $v \in S_{2,\Delta}$, (4.11), (3.4), and (3.9) imply

$$B_\Delta(u,v) = -\sum_{j=1}^{N-1} d_j[a_j\gamma_3^{[j]}h_j + a_{j+1}\gamma_2^{[j+1]}h_{j+1}] - \sum_{j=1}^{N-1} \epsilon d_j[a_{j+1}\gamma_5^{[j+1]} - a_j\gamma_6^{[j]}] \quad , \tag{4.12}$$

$$\text{(4.13) a)} \quad ||v||^p_{H^2_{p,\varepsilon,\Delta}} = \sum_{j=1}^{N} |a_j|^p \gamma_4^{[j]} h_j + \sum_{j=1}^{N-1} \varepsilon^p |a_{j+1}\gamma_5^{[j+1]} - a_j\gamma_6^{[j]}|^p \rho_j^{1-p} \quad ,$$

$$\text{b)} \quad \sum_{j=1}^{N} a_j\gamma_1^{[j]}h_j = 0 \quad .$$

In the argument that follows we take $p = 2$, $h_j = h$ for $j=1,\ldots N$, where $h = \frac{1}{N}$, and $\gamma_i^{[j]} = \gamma_i(h,\epsilon)$, $j = 1,\ldots,N$, depends in general on h and ϵ . Before elaborating on the next theorem, we will prove some lemmas.

Lemma 4.4. Let a_j, $j = 1,\ldots,N$, be such that

$$a_j - a_{j+1}K = 1, \quad j = 1,\ldots,N-1 \; , \tag{4.14}$$

and

$$\sum_{j=1}^{N} a_j = 0 \quad , \tag{4.15}$$

then

$$(4.16) \quad \text{(a)} \quad \Phi(K,N) = \sum_{j=1}^{N} a_j^2 = -\frac{N}{(1-K)^2} + N^2 \frac{K^N+1}{(K^N-1)(K^2-1)} , \quad K \neq 1 ,$$

and

$$\text{(b)} \quad \Phi(1,N) = \sum_{j=1}^{N} a_j^2 = \frac{N^3-N}{12} .$$

Proof. From (4.14) it follows that

$$a_j = \frac{1}{1-K} + AK^{-j+1}, \quad j = 1,\dots,N , \quad K \neq 1 .$$

Using (4.15) we find that

$$A = \frac{NK^{N-1}}{K^N-1} ,$$

and (4.16) follows easily.

If $K = 1$, $a_j = \frac{N-1}{2} - j$ which implies 4.16(b) .

Lemma 4.5. Let $\Phi(K,N)$ be defined by (4.16), then

$$(4.17) \qquad \Phi(K,N) \geq \frac{N(N-1)}{(1+|K|)^2} .$$

Proof. First suppose $K \geq 0$ $(K \neq 1)$. Then

$$\Phi(K,N) = -\frac{N}{(1-K)^2}\left[1 - \frac{N(1+K^N)}{(1-K)(1+K+\ldots+K^{N-1})}\right] =$$

$$= N\,\frac{N-1 + K(2N-4) +\ldots+ K^{N-2}(N-1)}{(1+K)(1+K+\ldots+K^{N-1})} .$$

Observing that $jN - j^2 \geq N-1$, for $j = 1,\ldots,N-1$, and since we are assuming that $K \geq 0$ we get

$$\Phi(K,N) \geq N\left[\frac{(N-1)(1+K+\ldots K^{N-2})}{(1+K)(1+\ldots+K^{N-1})}\right] \geq \frac{N(N-1)}{(1+K)^2} .$$

If $K = 1$ (4.17) follows directly from (4.16b). Next suppose that $K < 0$. If N is even, from (4.16) we obtain

$$\Phi(K,N) \geq \frac{N^2}{|K^2-1|} - \frac{N}{(1-K)^2} \geq \frac{N(N-1)}{(1+|K|)^2} .$$

If N is odd, then we have

$$\Phi(K,N) = -\frac{N}{(1-K)^2} + \frac{N^2(K^{N-1}-K^{N-2}+\ldots+1)}{(K^N-1)(K-1)} = -\frac{N}{(1-K)^2} + \frac{N^2(K^{N-1}-K^{N-2}+\ldots+1)}{(K-1)^2(K^{N-1}+K^{N-2}+\ldots+1)} .$$

Because N is odd and $K < 0$, we have

$$K^{N-1}-K^{N-2}+\ldots+1 \geq K^{N-1}+\ldots+1 > 0 ,$$

and so

$$\Phi(K,N) \geq \frac{N^2}{(K-1)^2} - \frac{N}{(1-K)^2} = \frac{N(N-1)}{(1+|K|)^2} ,$$

which completes the proof of the lemma.

Define ν and μ by

$$(4.19) \qquad \nu = \gamma_3 - \frac{\varepsilon}{h}\gamma_6 , \qquad \mu = \gamma_2 + \frac{\varepsilon}{h}\gamma_5 , \text{ and let } K = -\frac{\mu}{\nu}$$

when $\nu \neq 0$.

Then we have, by (4.12),

$$(4.20) \qquad B_\Lambda(u,v) = -h\sum_{j=1}^{N-1} d_j[a_j\nu + a_{j+1}\mu] = -h\nu\sum_{j=1}^{N-1} d_j[a_j - a_{j+1}K], \text{ provided } \nu \neq 0.$$

Furthermore, using (4.13) we have

$$(4.21) \qquad ||v||^2_{H^2_{2,\varepsilon,\Delta}} \geq h\gamma_4 \sum_{j=1}^{N} a_j^2 \quad .$$

Select v_o, where v_o is of the form (4.10), such that it is characterized by $\{a_j\}_{j=1}^N$ of Lemma 4.4. Then (4.20) and (4.14) imply that

$$(4.22) \qquad \sup_{\substack{||u||_{\overset{o}{H}_{2,\Delta}} \leq 1 \\ u\in S_{1,\Lambda}}} |B_\Lambda(u,v_o)| = \sup_{\substack{||u||_{\overset{o}{H}_{2,\Delta}} \leq 1 \\ u\in S_{1,\Lambda}}} |h||\nu|\sum_{j=1}^{N-1} d_j \leq |h|^{\frac{1}{2}}|\nu|\left(\sum_{j=1}^{N-1} d_j^2\right)^{1/2} \leq |\nu| \quad ,$$

where we used the fact that

$$(4.23) \qquad ||u||^2_{\overset{o}{H}_{2,\Delta}} \geq h\sum_{i=1}^{N-1} d_j^2 \quad .$$

Using (4.21), (4.22), and Lemmas (4.4) and (4.5), we obtain

$$(4.24) \qquad \inf_{\substack{\|v\|_{H^2_{2,\varepsilon,\Lambda}} \le 1 \\ u \in S_{2,\Lambda}}} \ \sup_{\substack{\|u\|_{H^o_o} \le 1 \\ v \in S_{2,\Lambda}}} |B_\Delta(u,v)| \le \frac{3|\nu|}{[\Phi(K,N)N^{-1}\gamma_4]^{\frac{1}{2}}} \le$$

$$\le \frac{3|\nu|(1+|K|)}{(N-1)^{\frac{1}{2}}\gamma_4^{\frac{1}{2}}} \le \frac{3(|\mu|+|\nu|)}{(N-1)^{\frac{1}{2}}\gamma_4^{\frac{1}{2}}} \quad .$$

A slight modification for $\nu = 0$ will yield the same result. Assume that $\phi_i \in H_2^1(0,1))$, $i = 1,\dots,m$, $m > 1$, are linearily independent, and

$$(4.25) \qquad \chi_j(x) = \sum_{i=1}^{n} c_i \phi_i\left(\frac{x - x_{j-1}}{h_j}\right) , \qquad x \in I_j \quad ,$$

where c_i depends in general on ε and h. We then have the following lemma.

Lemma 4.7. Let $\bar{c} = (c_1,\dots c_m)$.

Assume $\frac{\varepsilon}{h} \le \eta$ is sufficiently small. Then

$$(4.26) \qquad \sup_{\bar{c}} \frac{(|\nu|^2+|\mu|^2)}{\gamma_4} \le L < \infty$$

where L is independent of ε and h .

Proof. First note that (4.25) and the definitions of ν, μ, and γ_4 (see (4.19) and (4.10) iv), imply that $|\nu|^2$, $|\mu|^2$, and γ_4 are all quadratic forms in $\bar{c} = (c_1,\dots c_m)$. Therefore we can restrict ourselves to the case where $\sum_{i=1}^{m} c_i^2 = 1$.

The definition of γ_3 , (see (4.10)iii) and rescaling imply that

$$\gamma_3 = \sum_{i=1}^{m} c_i \int_0^1 [\frac{\varepsilon}{h}\phi_i'(y)+\phi_i(y)]\cdot y\, dy \le \max(1,\frac{\varepsilon}{h}) \sum_{i=1}^{m} |c_i|\ ||\phi_i||_{1,2,I} \quad .$$

Also $\gamma_6 = \chi_j(x_j) \le \sqrt{2} \sum_{i=1}^{m} |c_i|\ ||\phi_i||_{1,2,I}$. Thus $\nu = \gamma_3 - \frac{\varepsilon}{h}\gamma_6 \le$ $\max(1,3\frac{\varepsilon}{h}) \sum_{i=1}^{m} |c_i|\ ||\phi_i||_{1,2,I}$. We obtain the same approximation for μ and so $|\nu|^2 + |\mu|^2 \le 2\max(1,9\frac{\varepsilon^2}{h^2}) \sum_{i=1}^{m} ||\phi_i||^2_{1,2,I}$.

Since it is assumed that $\frac{\varepsilon}{h}$ is bounded, $|\nu|^2 + |\mu|^2$ is bounded as well, independent of ε and h .

Because $\{\phi_i\}_{i=1}^m$ are linearly independent in $L_2(I)$ and $\phi_i' \in L_2(I)$ $i=1,\ldots,m$, there exists an $\eta > 0$ such that $\frac{\varepsilon}{h} \le \eta$ implies that $\{\phi_i + \frac{\varepsilon}{h}\phi_i'\}_{i=1}^m$ are also linearly independent in $L_2(I)$. The only way for the left hand side of (4.26) to go to infinity would be for γ_4 to go to zero. That is, there must exist a sequence $\bar{c}_\ell$ such that $\gamma_4(\bar{c}_\ell) \to 0$ as $\ell \to \infty$. But $|\bar{c}_\ell| = 1$ and thus $\{\bar{c}_\ell\}$ has a subsequence converging to say $\bar{c}_o$. Since γ_4 depends continuously on $\bar{c}$, $\gamma_4(\bar{c}_o) = 0$, contradicting the linear independence of $\{\phi_i + \frac{\varepsilon}{h}\phi_i'\}_{i=1}^m$, and thus concluding the proof.

Now we can formulate the main result of this section as the following theorem.

Theorem 4.8. Let $h_j = h$, $j = 1,\ldots,N$, and $\frac{\varepsilon}{h} \le \eta$ be sufficiently small. Let $S_{2,\Delta} \in \overset{\circ}{H}{}^1_2$ be an N-1 dimensional space, such that if $v \in S_{2,\Delta}$, then

$$v'|_{I_j} = a_j\chi_j(x) \quad \text{with}$$

$$\chi_j(x) = \sum_{i=1}^{m} c_i\phi_i(\frac{x-x_{j-1}}{h_j}) \ , \quad x \in I_j \text{ , where}$$

$\{\phi_i\}_{i=1}^m$ are linearly independent in $L_2(I)$. Then

$$\inf_{\substack{\|v\|_{H^2_{2,\varepsilon,\Delta}}=1 \\ v\in S_{2,\Delta}}} \sup_{\substack{\|u\|_{\overset{o}{H}_\Delta}=1 \\ u\in S_{1,\Delta}}} \le KN^{-\frac{1}{2}}, \quad K>0 \text{ (independent of } \varepsilon \text{ and } h) .$$

Proof. The theorem follows immediately from (4.24) and Lemma 4.7.

It is readily seen that the space $S^{\alpha}_{2,\Delta}$ satisfies the assumptions of Theorem 4.8 for any α . Theorem 4.8 shows that the use of test functions recommended in [5], [10]-[12], [15]-[17], leads to an inf-sup condition converging to 0 as $N \to \infty$, when $\frac{\varepsilon}{h}$ is held constant. Moreover, there is no choice of $\phi_i(x)$, independent of ε and h , which would lead to quasi-optimal results. Nevertheless, in the next section we will show that the situation significantly improves when assumptions about the input data are made.

5. MODIFIED VARIATIONAL PRINCIPLE

In this section we perform a more detailed analysis of the subspace $S^{\alpha}_{2,\Delta}$, defined by (4.3). In particular, we will compare this space to $S^{E}_{2,\Delta}$, when each is being tested against the subspace $S_{1,\Delta}$ of piecewise linear trial functions.

To distinguish between the spaces $S^{E}_{2,\Delta}$ and $S^{\alpha}_{2,\Delta}$, we will write v_E and v_{α}, denoting a function in $S^{E}_{2,\Delta}$ and $S^{\alpha}_{2,\Delta}$ respectively. By straightforward computations we get

$$(5.1) \qquad v'_E(x)\big|_{I_j} = \frac{v_j - v_{j-1}}{h_j} E_j(x) \quad ,$$

where

$$(5.2) \qquad E_j(x) = \frac{h_j}{\varepsilon} \frac{e^{\frac{x_{j-1}-x}{\varepsilon}}}{(1-e^{-h_j/\varepsilon})} \quad , \quad v_j = v_E(x_j) \quad .$$

Analogously,

$$(5.3) \qquad v'_{\alpha}(x)\big|_{I_j} = \frac{v_j - v_{j-1}}{h_j} \chi_j(x) \quad ,$$

where

$$(5.4) \qquad \chi_j(x) = 1 - \frac{6\alpha_j}{h_j}(x - x_{j-\frac{1}{2}}) \quad , \quad x_{j-\frac{1}{2}} = \frac{x_j + x_{j-1}}{2}$$

and once again $v_j = v_{\alpha}(x_j)$.

Let P_j be the orthogonal projection operator from $L_2(I_j)$ to linear functions in I_j . By direct computation we obtain

$$(5.5)\qquad P_j E_j(x) = 1 - \frac{6}{h_j}\left(\coth \frac{h_j}{2\varepsilon} - \frac{2\varepsilon}{h_j}\right)(x - x_{j-1}) = 1 - \frac{6}{h_j}\, \alpha_j^o (x - x_{j-\frac{1}{2}}) = \chi_j^o(x) ,$$

where

$$(5.6)\qquad \alpha_j^o = \alpha_j^o(\varepsilon, h_j) = \coth \frac{h_j}{2\varepsilon} - \frac{2\varepsilon}{h_j} .$$

From now on we denote $S_{2,\Delta}^{\alpha} = S_{2,\Delta}^{\alpha_o}$, that is, we consider the particular $\alpha = \alpha^o$, where α_j^o is defined by (5.6). We remark that α^o is the same as the "optimal" α described in [11], [12], [15]-[17].

We now present some important relationships between $S_{2,\Delta}^{E}$ and $S_{2,\Delta}^{\alpha}$.

Lemma 5.1. Let $v_E \in S_{2,\Delta}^{E}$ and $v_\alpha \in S_{2,\Delta}^{\alpha}$, $(\alpha = \alpha^o)$, be such that $v_E(x_j) = v_\alpha(x_j)$ for $j = 0,\ldots,N$. Then we have

$$(5.7)\qquad (v_E', z)_I = (v_\alpha', z)_I$$

for any $z \in L_2(I)$, z linear on I_j for $j = 1,\ldots,N$, (z not necessarily continuous at x_j), and

$$(5.8)\qquad (v_E, q)_I = (v_\alpha, q)_I$$

for any $q \in L_2(I)$, q constant on I_j for $j = 1,\ldots,N$.

Proof. (5.7) follows immediately from (5.1)-(5.6). To prove (5.8) let q be piecewise constant on Δ. Let $z(x) = \int_o^x q(t)dt$. Then z is linear on I_j and therefore by (5.7) we have

$$\int_0^1 v_E' z dx = \int_0^1 v_\alpha' z dx \ .$$

Integration by parts, recalling that z is continuous, and that $v_E(0) = v_\alpha(0)$ and $v_E(1) = v_\alpha(1)$, yields the desired result.

Theorem 5.2. Let $u \in S_{1,\Delta}$, v_E, v_α be as in lemma 5.1. Then

$$B_\Delta(u,v_E) = B_\Delta(u,v_\alpha) \quad . \tag{5.9}$$

Furthermore, let f be a linear combination of Dirac "delta functions" centered at the mesh points x_j, $j = 1,\ldots,N-1$, and any piecewise constant function on Δ. Let $u_E \in S_{1,\Delta}$ and $u_\alpha \in S_{1,\Delta}$ be such that

$$B_\Delta(u_E,v_E) = (f,v_E) \ , \quad \forall v_E \in S^E_{2,}$$

and

$$B_\Delta(u_\alpha,v_\alpha) = (f,v_\alpha) \ , \quad \forall v_\alpha \in S^\alpha_{2,\Delta} \ .$$

Then

$$u_E = u_\alpha \quad . \tag{5.10}$$

Proof. For $u \in S_{1,\Delta}$ and $v \in S^E_{2,\Delta}$ or $v \in S^\alpha_{2,\Delta}$ we have $B_\Delta(u,v) = (\varepsilon u'-u,v')$. Since $\varepsilon u' - u$ is linear on I_j, (5.7) implies (5.9).

The fact that $v_E(x_j) = v_\alpha(x_j)$ for $j = 1,\ldots,N-1$, and (5.8) together with (5.9), yield (5.10).

Theorem 5.2 shows that the global stiffness matrices generated by using $S_{1,\Delta}$ with $S^E_{2,\Delta}$ and $S^\alpha_{2,\Delta}$, $(\alpha=\alpha_o)$, respectively, are identical. However, we have shown (see

Theorem 4.2), that for any $u \in \overset{o}{H}_{p,\Delta}$

(5.11) $$||u-u_E||_{\overset{o}{H}_{p,\Delta}} \leq C \inf_{w \in S_{1,\Delta}} ||u-w||_{\overset{o}{H}_{p,\Delta}}$$

with C independent of Δ, ε. On the other hand, we have shown (see Theorem 4.8) that there exists $u \in \overset{o}{H}_{2,\Delta}$, (resp. $f \in (H^2_{2,\varepsilon,\Delta})'$), such that

(5.12) $$||u-u_\alpha||_{\overset{o}{H}_{2,\Delta}} \geq Ch^{-\frac{1}{2}} \inf_{w \in S_{1,\Delta}} ||u-w||_{\overset{o}{H}_{2,\Delta}}$$

on a uniform partition. Comparing (5.11) and (5.12) we see that the use of $S^\alpha_{2,\Delta}$ will deteriorate the result for some u, (resp. f). The question arises as to whether or not this f has any "engineering meaning", that is, will this effect be observed in practice.

The preceding theorem says that if f is piecewise constant, then $u_E = u_\alpha$. This suggests that if f is "nearly" piecewise constant, then u_α may be "nearly" equal to u_E. We shall now show that this is true; that is, if f (the input data) is "reasonable" then the term $Ch^{-\frac{1}{2}}$ can be replaced by C independent of ε and h.

Theorem 5.3. Let $f \in (H^2_{q,\varepsilon,\Delta})'$, $f \in H^1_\infty(I_j)$ $j = 1,\ldots,N$. Denote by $u \in \overset{o}{H}_{p,\Delta}$ the exact weak solution of the problem

$$-\varepsilon u'' + u' = f \ ,$$

$$u(0) = u(1) = 0 \ ,$$

that is,

$$B_\Delta(u,v) = (f,v), \quad \forall v \in H^2_{q,\varepsilon,\Delta} . \tag{5.13}$$

Let $u_\alpha \in S_{1,\Delta}$ be such that

$$B_\Delta(u_\alpha,v) = (f,v) , \quad \forall v \in S^\alpha_{2,\Delta} , \qquad \alpha = \alpha^o . \tag{5.14}$$

Then for $1 \le p \le \infty$,

$$||u-u_\alpha||_{\overset{o}{H}_{p,\Delta}} \le (1+2^{1/p}) \inf_{w \in S_{1,\Delta}} ||u-w||_{\overset{o}{H}_{p,\Delta}} + \tag{5.15}$$

$$+ C \max_{j=1,\ldots,N} \left[h_j^2 ||f||_{H^1_\infty(I_j)} \right] ,$$

with C independent of Δ , ε , p , and f .

We first prove a lemma.

Lemma 5.4.

$$||v'_\alpha - v'_E||_{L_q(I_j)} \le C||v'_E||_{L_q(I_j)} , \qquad 1 \le q \le \infty$$

with C independent of q , Δ, and ε.

Proof. Obviously it is sufficient to show that

$$||v'_\alpha||_{L_q(I_j)} \le C||v'_E||_{L_q(I_q)} . \tag{5.16}$$

Since v'_α is the projection of v'_E onto piecewise linear functions, we have

(5.17) $$(v'_E - v'_\alpha, v'_\alpha)_{I_j} = 0 ,$$

and hence

$$\int_{I_j} (v'_\alpha)^2 dx = \int_{I_j} v'_E v'_\alpha dx \le ||v'_E||_{L_q(I_j)} ||v'_\alpha||_{L_p(I_j)} .$$

Therefore,

(5.18) $$\frac{\left[\int_{I_j} (v'_\alpha)^2 dx\right]^q}{\left[\int_{I_j} |v'_\alpha|^p dx\right]^{\frac{q}{p}}} \le \int_{I_j} |v'_E|^q dx , \quad 1 \le q < \infty .$$

(Where, if $q = 1$, we replace the denominator by $||v'_\alpha||_{L_\infty(I_j)}$.)

Returning to (5.1) and (5.3), we see that instead of v'_α and v'_E, we can deal with $\chi_j(x)$ and $E_j(x)$, respectively. Since

$$\chi_j(x) \le 1 + 3\alpha_j , \qquad x \in I_j ,$$

and

$$\chi_j(x) \ge 1 , \qquad x_{j-1} \le x \le x_{j-\frac{1}{2}} ,$$

we get

$$\left(\int_{I_j} \chi_j^2(x) dx\right)^q \ge \left(\frac{h_j}{2}\right)^q ,$$

and

$$\left(\int_{I_j} |\chi_j|^p dx\right)^{\frac{q}{p}} \leq (1+3\alpha_j)^q h_j^{\frac{q}{p}} \quad .$$

Therefore,

$$(5.19) \qquad \frac{\left[\int_{I_j} (\chi_j)^2 dx\right]^q}{\left[\int_{I_j} |\chi_j|^p dx\right]^{\frac{q}{p}}} \geq h_j^{q - \frac{q}{p}} (1+3\alpha_j)^{-q} 2^{-q} = h_j (2(1+3\alpha_j))^{-q} \quad .$$

However,

$$\int_{I_j} |\chi_j|^q dx \leq h_j (1+3\alpha_j)^q \quad ,$$

which implies

$$(5.20) \qquad \int_{I_j} |E|^q dx \geq \frac{\left[\int_{I_j} \chi^2(x) dx\right]^q}{\left[\int_{I_j} |\chi(x)|^p dx\right]^{\frac{q}{p}}} \geq h_j ((1+3\alpha_j) 2)^{-q}$$

$$\geq 2^{-q} (1+3\alpha_j)^{-2q} \int_{I_j} |\chi_j|^q dx \quad .$$

Therefore, by (5.18) - (5.20) we have

$$(5.21) \qquad ||v'_\alpha||_{L_q(I_j)} \leq 2(1+3\alpha_j)^2 ||v'_E||_{L_q(I_j)} \quad .$$

Furthermore, it is easy to see from (5.6) that $0 \leq \alpha \leq 1$, and thus because of (5.21), (5.16) holds for $1 \leq q < \infty$. If $q = \infty$ we simply note that $||E_j||_{L\infty(I_j)} \leq 1 + 3\alpha_j$ and hence (5.16) also holds in this case. Now we will prove Theorem 4.3.

Proof. Let $v_E(x_j) = v_\alpha(x_j)$ for $j = 0,\ldots N$.

Since

$$(f,v_\alpha) = B_\Delta(u_\alpha,v_\alpha) = B_\Delta(u_\alpha,v_E) ,$$

and

$$(f,v_E) = B_\Delta(u_E,v_E) ,$$

we get

$$B(u_E-u_\alpha,v_E) = (f,v_E-v_\alpha) .$$

Hence, applying Theorem 4.1 yields

$$||u_E-u_\alpha||_{H^o_{p,\Delta}} \leq C \sup_{v_E \in S^E_{2,\Delta}} \frac{(f,v_E-v_\alpha)}{||v_E||_{H^2_{q,\varepsilon,\Delta}}} .$$

By Theorems 4.2, 4.1, 2.2, and the triangle inequality

$$(5.22)\qquad ||u-u_\alpha||_{H^o_{p,\Delta}} \le ||u-u_E||_{H^o_{p,\Delta}} + ||u_E-u_\alpha||_{H^o_{p,\Delta}} \le$$

$$\le (1+2^{1/p}) \inf_{w\in S_{1,}} ||u-w||_{H^o_{p,\Delta}} + C \sup_{v_E\in S^E_{2,\Delta}} \frac{(f,v_E-v_\alpha)}{||v_E||_{H^2_{q,\varepsilon,\Delta}}}$$

Therefore, we have to deal with the term

$$\sup_{v_E\in S^E_{2,\Delta}} \frac{(f,v_E-v_\alpha)}{||v_E||_{H^2_{q,\varepsilon,\Delta}}} \quad .$$

Denote by $\tilde{f}$ the function which is constant on I_j, which satisfies $\int_{I_j} (f-\tilde{f})dx = 0$.

If we let $F_j(x) = \int_{x_{j-1}}^{x} (f-\tilde{f})(t)dt$, then $F_j(x_{j-1}) = F_j(x_j) = 0$, *and since* $f\in H^1_\infty(I_j)$, we have

$$(5.23)\qquad |F_j(x)| \le \frac{1}{8} h_j^2 ||f||_{1,\infty,I_j} \quad .$$

Further, we denote by $F(x)$ the function such that $F(x)|_{I_j} = F_j(x)$. By Lemma 5.1 we get

$$(\tilde{f},v_E-v_\alpha) = 0 \quad ,$$

and hence

$$(f,v_E-v_\alpha) = (f-\tilde{f},v_E-v_\alpha) = (F',v_E-v_\alpha) \quad .$$

Integration by parts, (5.23), Lemma 5.4 and Lemma 3.2 yield

$$|(f,v_E-v_\alpha)| \le \sum_{j=1}^{N} |(F_j,v_E'-v_\alpha')_{I_j}| \le \sum_{j=1}^{N} \frac{1}{8} h_j^2 ||f||_{1,\infty,I_j} ||v_E'-v_\alpha'||_{L_1(I_j)} \le$$

$$\le C \max_{j=1,\ldots,N} \left[\frac{1}{8} h_j^2 ||f||_{1,\infty,I_j}\right] ||v_E||_{H^2_{1,\varepsilon,\Delta}} .$$

It can be easily verified from (3.9) and (4.2) that

$$||v_E||_{H^2_{1,\varepsilon,\Delta}} \le 2^{\frac{1}{q}} ||v_E||_{H^2_{q,\varepsilon,\Delta}} , \quad 1 \le q \le \infty .$$

Thus, we have

(5.24) $$\sup_{v_E \in S^E_{2,\Delta}} \frac{|(f,v_E-v_\alpha)|}{||v||_{H^2_{q,\varepsilon,\Delta}}} \le C \max_j \left[h_j^2 ||f||_{1,\infty,I_j}\right] ,$$

with C independent of Δ, q, ε and f. (5.24) together with (5.22) proves the theorem.

6. NUMERICAL CONSIDERATIONS

Theorem 5.3 shows that for general meshes and parabolically upwinded test functions, the error consists of two parts. In this section we comment on the interplay between them.

For the error estimate (5.15) to be "nearly" quasi-optimal, the second term

$$(6.1) \qquad C \max_{j=1,\dots N} [h_j^2 ||f||_{H^1_\infty(I_j)}] \ ,$$

should be smaller than, or of the same magnitude as, the first (quasi-optimal) term

$$(6.2) \qquad (1+2^{1/p}) \inf_{w \in S_{1,\Delta}} ||u-w||_{H^0_{p,\Delta}} \ .$$

It is well known from singular perturbation theory that the exact solution to (1.1) can be written as a combination of a smooth term and a boundary layer term. The boundary layer term is of the form $e^{(x-1)/\varepsilon}$; thus, it is sufficient to analyze

$$\inf_{w \in S_{1,\Delta}} || e^{(x-1)/\varepsilon} - w ||_{H^0_{p,\Delta}} \ .$$

A simple analysis of this term on a uniform mesh with stepsize h implies that for $\varepsilon \ll h$

$$\inf_{w \in S_{1,\Delta}} ||u-w||_{H^0_{p,\Delta}} \geq Kh^{1/p} \ ,$$

with K independent of ε, h and p . Thus, (6.1) can be neglected when compared to (6.2).

If $\epsilon >> h$, standard approximation theory says that (6.2) will have magnitude h^2, the same magnitude as (6.1).

Finally, we remark that the constant C in (6.1) can be computed exactly. In a forthcoming paper, an adaptive approach with a-posteriori error estimates will be studied.

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